NON-POLYNOMIAL LAGRANGIANS IN THE SKYRME MODEL

Jorge Ananias Neto
Departamento de Física, ICE
Universidade Federal de Juiz de Fora 36036-330
Juiz de Fora, MG, Brazil

Abstract

We choose three different coupling constants for a particular higher-derivative term in the Skyrme model that allows the total Lagrangian to converge in a binomial, geometric and a logarithmic form. Improved numerical results are obtained.

PACS number: 12.40.-y

The Skyrme model[1] consists of treating baryons as soliton solutions in the non-linear Sigma model with an additional stabilizer Skyrme term. We have recently studied [2] the introduction of another four derivative stabilizer term given by

$$L_2 = \int d^3r\, c_2 \left[Tr\left(\partial_\mu U \partial^\mu U^+\right)\right]^2 , \tag{1}$$

where $c_2 \equiv \frac{1}{32e^2}$. This form, known as the symmetric Skyrme term, is the square of the chiral SU(2) Sigma Model whose original Lagrangian is

$$L_1 = \frac{F_\pi^2}{16} \int d^3r\, Tr\left(\partial_\mu U \partial^\mu U^+\right) , \tag{2}$$

where F_π is the pion decay constant. Using the stabilizer L_2 term as a pattern for the inclusion of the higher-order derivative terms, we have verified an improvement in the physical values given by Skyrme Model. The purpose of this paper is to perform a study about the possibility of choosing different parameter relations in the higher derivative terms, consequently leading to a new types of Skyrme-like Lagrangian.

Thus, the standard form of the Lagrangian terms is

$$L_n = \int d^3r\, c_n \left[Tr\left(\partial_\mu U \partial^\mu U^+\right)\right]^n , \tag{3}$$

where n=1,2,... . Performing the usual collective coordinate expansion[3] $U(r,t) = A(t)U_0(r)A^+(t)$, being A a SU(2) matrix[1], we can write the Lagrangian (3) in the form

$$L_n = \int d^3r\, c_n \left[-2M + I\, Sp\right]^n , \tag{4}$$

where M is given by $\mathrm{M}^2 \equiv \left[\frac{2\sin^2 F(r)}{r^2} + F'^2(r)\right]$, the inertia moment I is written as $I \equiv \frac{8}{3}\sin^2 F$ and Sp is given by $Sp \equiv Tr\left[\partial_0 A \partial_0 A^{-1}\right]$. We can read the final form of the Lagrangian containing all derivative terms as

$$\begin{aligned} L = -c_1 \textstyle\int d^3r\ [2M - I\, Sp] - c_2 \int d^3r\ [2M - I\, Sp]^2 \\ \ldots - c_n \textstyle\int d^3r\ [2M - I\, Sp]^n . \end{aligned} \tag{5}$$

[1] Consequently, the matrix A can be written as $A = a_0 + i\tau_j a_j$.

[2]Here we have used the hedgehog ansatz, $U = \exp\left(i\tau.\hat{r}F(r)\right)$, where F(r) is called the chiral angle.

This arrangement, as we will see in the next sections, ensures the positivity of the total Hamiltonian. Then, the idea of this work consists in choosing different parameter relations that permit to sum the Lagrangian with higher derivative terms in a well known specific form. We set three different coefficient relations [4]: a) $K_n \equiv \frac{c_n}{c_1} = \begin{pmatrix} s \\ n-1 \end{pmatrix} /(2e^2F_\pi^2)^{n-1},\ n = 1,2,3\dots,$ which permits the total Lagrangian to converge to a binomial form; b) $K_n \equiv \frac{c_n}{c_1} = \frac{1}{(2e^2F_\pi^2)^{n-1}}$ $, n = 1,2,3\dots,$ in which the total Lagrangian is summed in a geometric series form; and c)$K_n \equiv \frac{c_n}{c_2} = \frac{(-1)^{n-1}}{(n-2)(2e^2F_\pi^2)^{n-2}}$ $, n = 3,4,5\dots,$ which permits the total Lagrangian to converge to a logarithmic form.

Binomial Form

If we set $K_n \equiv \frac{c_n}{c_1} = \begin{pmatrix} s \\ n-1 \end{pmatrix} /(2e^2F_\pi^2)^{n-1}$ the total Lagrangian (5) converges in a binomial form when $n \to \infty$,

$$\begin{aligned} L = -c_1 \textstyle\int d^3r\ [2M - I\,Sp]\,[1 + \frac{c_2}{c_1}\int d^3r\ [2M - I\,Sp] \\ \dots + \textstyle\frac{c_n}{c_1}\int d^3r\ [2M - I\,Sp]^{n-1}] \\ = -c_1 \textstyle\int d^3r[2M - I\,Sp][1 + \frac{2M - I\,Sp}{2e^2F_\pi^2}]^s\,, \end{aligned} \tag{6}$$

where $c_1 \equiv \frac{F_\pi^2}{16}$. In the process of collective coordinates quantization it is necessary that in the Lagrangian (6) we have only a linear term in *Sp*. Then, making a Taylor series expansion and retaining only the linear term in *Sp*, we obtain an expression for the Hamiltonian [3], written as

$$H = M_T + \frac{1}{8I_T}\sum_{i=0}^{3}\pi_i^2\ , \tag{7}$$

where

$$M_T = \frac{F_\pi}{e}\frac{\pi}{2}\int_0^\infty dx x^2 M\,[1 + M]^s\ , \tag{8}$$

[3] The Hamiltonian H is defined by $H = \pi_i\dot{a}_i - L$, where $\pi_i = \frac{\partial L}{\partial \dot{a}_i}$.

Figure 1: Behavior of the parameter B defined by $B \equiv x^2 F(x)$, where F(x) is the numerical variational solution of the classical binomial Hamiltonian form including terms up to power s=3/2, s=5/2, s=7/2, s=11/2, and s=21/2.

and

$$I_T = \frac{2\pi}{3}\frac{1}{e^3 F_\pi}\int_0^\infty dx x^2 \sin^2 F \left[s\, M\, (1+M)^{s-1} + (1+M)^s\right] . \tag{9}$$

In the last expression we have used the dimensionless variable $x = eF_\pi r$.

The quantized Hamiltonian form is obtained taking $\pi_i = -i\frac{\partial}{\partial a_i}$, which leads to

$$H = M_T + \frac{1}{8I_T}\left(-\frac{\partial^2}{\partial a_i^2}\right) = M_T + \frac{l(l+2)}{8I_T} , \quad l = 1, 2, 3, \dots \; . \tag{10}$$

The numerical solution of F(x) with the boundary conditions $F(0) = \pi$ and $F(\infty) = 0$, for different values of s powers, is obtained using the variational Euler-lagrange equation which is given by

$$
\begin{aligned}
&[8x^2F'^2sW^{(s-1)} + 4x^2F'^2s(s-1)(W-1)W^{(s-2)} + 2x^2W^s \\
&+2x^2s(W-1)W^{(s-1)}]F'' + 4xW^sF' + 2x^2F'sW^{s-1}dax \\
&+4xF'(W-1)sW^{(s-1)} + 2x^2F'sW^{(s-1)}dax \\
&+2x^2F's(s-1)(W-1)W^{(s-2)}dax \\
&-x^2W^sdaf - x^2(W-1)sW^{(s-1)}daf = 0 \ ,
\end{aligned} \tag{11}
$$

where $W \equiv [1 + \frac{2\sin^2 F}{x^2} + F'^2]$, dax $\equiv [\frac{2\sin 2FF'}{x^2} - \frac{4\sin^2 F}{x^3}]$ and daf $\equiv [\frac{2\sin 2F}{x^2}]$. The soliton solution is obtained if we impose the condition, $\lim_{x\to\infty} F(x) = \frac{B}{x^2}$. In figure 1, we show the numerical behavior of parameter B, which is directly proportional to the axial vector coupling constant, $g_A = \frac{2\pi}{3}\frac{B}{e^2}$.

Using the masses of the Nucleon ($M_N = 939 Mev$) and of the Delta ($M_\triangle = 1232 Mev$) as input parameters, we determine the pion decay constant F_π and the dimensionless Skyrme parameter e. The main physical results are shown in table 1, according to ref.[3],

TABLE 1- Physical parameters in the Skyrme Model
(binomial form)

s	3/2	5/2	7/2	11/2	21/2	ANW	expt.
$F_\pi(Mev)$	136	140	142	143	143	129	186
e	6.81	9.65	11.81	15.29	21.55	5.45	-
$< r^2 >^{\frac{1}{2}}_{I=0}\ (fm)$	0.60	0.60	0.61	0.61	0.61	0.59	0.72
μ_p	1.74	1.74	1.74	1.74	1.75	1.87	2.79
μ_n	-1.22	-1.21	-1.21	-1.21	-1.20	-1.33	-1.91
g_A	0.73	0.76	0.77	0.78	0.79	0.61	1.23

where ANW are the results of Adkins, Nappi and Witten (ref. [3]). These results indicate a convergence of the physical parameters to stable values with the increase of parameter s.

Geometric Form

If we set $K_n \equiv \frac{c_n}{c_1} = \frac{1}{(2e^2F_\pi^2)^{n-1}}$ the total Lagrangian (5) converges in a geometric form when $n \to \infty$,

$$\begin{aligned} L = -c_1 \int d^3r\ [2M - I\,Sp]\,[1 + \tfrac{c_2}{c_1} \int d^3r\ [2M - I\,Sp] \\ \ldots + \tfrac{c_n}{c_1} \int d^3r\ [2M - I\,Sp]^{n-1}] \\ = -c_1 \int d^3r\,[2M - I\,Sp]\left[1 - \tfrac{[2M - I\,Sp]}{2e^2F_\pi^2}\right]^{-1}\ , \end{aligned} \tag{12}$$

where $c_1 \equiv \frac{F_\pi^2}{16}$. Using the same procedure adopted in the binomial section, i.e., retaining only the linear term *Sp* in the Lagrangian (12), we can write the eigenvalues of the quantized Hamiltonian as

$$H = M_T + \frac{l(l+2)}{8I_T}\ , \quad l = 1, 2, 3 \ldots, \tag{13}$$

where

$$M_T = \frac{F_\pi}{e}\frac{\pi}{2}\int_0^\infty dx x^2 M\,[1 - M]^{-1}\ , \tag{14}$$

and

$$I_T = \frac{2\pi}{3}\frac{1}{e^3F_\pi}\int_0^\infty dx x^2 \sin^2 F\left[M\,(1 - M)^{-2} + (1 - M)^{-1}\right]\ . \tag{15}$$

The Euler-Lagrange equation which gives the numerical solution of F(x) with the boundary conditions $F(0) = \pi$ and $F(\infty) = 0$, is given by,

$$\begin{aligned} \left[2x^2 + 8x^2F'^2\,Wg^{-1}\right]F'' + 4xF' + 4x^2F'Wg^{-1}dax \\ -x^2daf = 0\ , \end{aligned} \tag{16}$$

where $Wg \equiv [1 - (\frac{2\sin^2 F}{x^2} + F'^2)]$, and dax and daf are defined in (11). In figure 2 we show the behavior of the soliton parameter B, and as we have

Figure 2: Behavior of the parameter B defined by $B \equiv x^2 F(x)$, where F(x) is the numerical variational solution of the classical geometric series Hamiltonian form.

observed in the binomial case, this parameter is directly proportional to the axial vector coupling constant, $g_A = \frac{2\pi}{3}\frac{B}{e^2}$.

Again, using the masses of the Nucleon and of the Delta as input parameters, we obtain the main physical results, which are shown in table 2.

TABLE 2- Physical parameters in the Skyrme Model

	geometric form	ANW	expt.
$F_\pi (Mev)$	152	129	186
e	8.48	5.45	-
$< r^2 >^{\frac{1}{2}}_{I=0} \, (fm)$	0.61	0.59	0.72
μ_p	1.75	1.87	2.79
μ_n	-1.21	-1.33	-1.91
g_A	0.84	0.61	1.23

Logarithmic Form

Defining $K_n \equiv \frac{c_n}{c_2} = \frac{(-1)^{n-1}}{(n-2)(2e^2F_\pi^2)^{n-2}}$, $n = 3, 4 \ldots$, the total Lagrangian (5) converges in a logarithmic specific form which admits a soliton solution when $n \to \infty$,

$$\begin{gathered}
L = -c_1 \int d^3r \; [2M - I\, Sp] - c_2 \int d^3r \; [2M - I\, Sp]^2 \, [1 \\
+ \tfrac{c_3}{c_2} \int d^3r \; [2M - I\, Sp] \ldots + \tfrac{c_n}{c_2} \int d^3r \; [2M - I\, Sp]^{n-1}] \\
= -c_1 \int d^3r \, [2M - I\, Sp] \\
-c_2 \int d^3r \, [2M - I\, Sp]^2 \left[1 + \log\left(1 + \tfrac{2M - I\, Sp}{(2e^2F_\pi^2)}\right)\right] \; ,
\end{gathered} \tag{17}$$

where $c_1 \equiv \frac{F_\pi^2}{16}$ and $c_2 \equiv \frac{1}{32e^2}$. The total quantized eigenvalues Hamiltonian are given by

$$H = M_T + \frac{l(l+2)}{8I_T} \, , \quad l = 1, 2, 3 \ldots , \tag{18}$$

where

$$M_T = \frac{F_\pi}{e} \frac{\pi}{2} \int_0^\infty dx x^2 M \left[1 + M \left(1 + \log\left(1 + M\right)\right)\right] \, , \tag{19}$$

and

$$\begin{aligned}
I_T = \frac{2\pi}{3} \frac{1}{e^3 F_\pi} \int_0^\infty dx x^2 \sin^2 F \; [\, 1 + M \, (M(1+M)^{-1} \\
+2(1 + \log(1 + M) \,)) \,] \; .
\end{aligned} \tag{20}$$

The Euler-Lagrange equation which gives the soliton solution with the boundary conditions $F(0) = \pi$ and $F(\infty) = 0$, is

$$\begin{gathered}
[2x^2 + 4x^2 M + 8x^2 F'^2 + 4x^2 F'^2 Sl + 2x^2 M Sl + 12x^2 F'^2 M (1+M)^{-1} \\
-4x^2 F'^2 M^2 (1+M)^{-2}] F'' + 4x F' + 8x F' M + 4x^2 F' dax + 4x F' M Sl \\
+2x^2 F' dax Sl + 6x^2 F' M (1+M)^{-1} dax - 2x^2 F' M^2 (1+M)^{-2} dax \\
-2 \sin(2F) - 2x^2 M daf - 2x^2 M daf \log(1+M) \\
-x^2 M^2 (1+M)^{-1} daf = 0 \;\; ,
\end{gathered} \tag{21}$$

Figure 3: Behavior of the parameter B defined by $B \equiv x^2 F(x)$, where F(x) is the numerical variational solution of the classical logarithmic series Hamiltonian form.

where $Sl \equiv [2\log(1+M) + M(1+M)^{-1}]$, and dax and daf are defined in (11). With the masses of the Nucleon and of the Delta as input parameters, the physical results (table 3) are given by

TABLE 3- Physical parameters in the Skyrme Model

	logarithmic form	ANW	expt.
$F_\pi (Mev)$	141	129	186
e	6.69	5.45	-
$< r^2 >^{\frac{1}{2}}_{I=0}\ (fm)$	0.60	0.59	0.72
μ_p	1.74	1.87	2.79
μ_n	-1.21	-1.33	-1.91
g_A	0.76	0.61	1.23

Before going to the final comments, we would like to mention that there are many controversies about the possibility of Lagrangian terms which contain quartic or more time derivatives destabilizing the Skyrmion [5]. At first, as we have described in the binomial section, this problem can be overcome with the linear procedure used in the collective coordinates expansion. Then, it is also interesting to study the process of quantization of collective coordinates including higher derivative time terms. This question is a natural step of continuation of this work and it will be object of a forthcoming paper.

The three Skyrme-type Lagrangians that are presented in this paper; that is, the binomial, geometric and logarithmic plus exponential form developed in [2] are an attempt to include, in a simple way, the contribution of a specific higher derivative term(3). We observe that the physical parameters which result from this procedure go in the right direction. It is interesting to point out that our binomial Skyrme form resembles the unconventional Born-Infeld electrodynamics worked out by Dirac [6]. Here we would like to mention that the best results are obtained using the geometric form. We hope that with the four different forms of the Skyrme Lagrangian we have covered the main contributions of particular higher derivative terms(known as the symmetric Skyrme term) in the usual physical values of the Skyrme model.

I would like to thank M.G. do Amaral for critical reading and the Departamento de Campos e Partículas of Centro Brasileiro de Pesquisas Físicas for hospitality while part of this work was carried out.

References

[1] T.H.Skyrme, Proc.Roy.Soc.A260, 127 (1961).

[2] J.A.Neto, J. Phys. G20, 1527 (1994).
S.Dubé and L.Marleau, Phys.Rev.D41,5, 1606 (1990).
G.S.Adkins, Chiral Solitons edited by Keh-Fei Liu (World Scientific) p99 (1987).

[3] G.S.Adkins,C.R. Nappi and E. Witten, Nucl.Phys.B228, 552 (1983).

[4] I.S.Gradshteyn and I.M.Ryzhik, Table of Integrals, Series and Products, Academic Press Inc.

[5] G.Kalbermann, J.M.Eisenberg and Andreas Schafer, Phys.Lett.B339, 211 (1994).

[6] P.A.M.Dirac, Proc.Roy.Soc.A257, 32 (1960).
S.Deser, M.J.Duff and C.J.Isham, Nucl.PhysB114, 29 (1976).